INCOMPRESSIBILITY OF QUADRATIC WEIL TRANSFER OF GENERALIZED SEVERI-BRAUER VARIETIES

NIKITA A. KARPENKO

ABSTRACT. Let X be the variety obtained by the Weil transfer with respect to a quadratic separable field extension of a generalized Severi-Brauer variety. We study (and, in some cases, determine) the canonical dimension, incompressibility, and motivic indecomposability of X. We determine the canonical 2-dimension of X (in the general case).

1. INTRODUCTION

The expression *canonical dimension* appeared for the first time in [1]. The p-local version (where p is a prime), called *canonical p-dimension*, comes from [12]. They both can be introduced as particular cases of a (formally) older notion of the *essential (p-)dimension* (although, in fact, the canonical dimension has been implicitly studied for a long time before). Below, we reproduce the modern definitions of [15].

A connected smooth complete variety X over a field F is called *incompressible*, if any rational map $X \dashrightarrow X$ is dominant.

In most cases when it is known that a particular variety X is incompressible, it is proved by establishing that X is *p-incompressible* for some positive prime integer p. This is a stronger property which says that for any integral variety X', admitting a dominant morphism to X of degree coprime with p, any morphism $X' \to X$ is dominant.

In some interesting cases, p-incompressibility of X is, in its turn, a consequence of a stronger property – indecomposability of the *p-motive* of X. Here by the p-motive we mean the classical Grothendieck motive of the variety constructed using the Chow group with coefficients in the finite field of p elements.

Canonical dimension $\operatorname{cdim} X$ is a numerical invariant which measures the level of the incompressibility. It is defined as the least dimension of the image of a rational map $X \dashrightarrow X$. *Canonical p-dimension*, the p-local version, measures the p-incompressibility and is the least dimension of the image of a morphism $X' \to X$, where X' runs over the integral varieties admitting a dominant morphism to X of a p-coprime degree. We always have the inequalities $\operatorname{cdim}_p X \leq \operatorname{cdim} X \leq \dim X$; the equality $\operatorname{cdim} X = \dim X$ means incompressibility and the equality $\operatorname{cdim}_p X = \dim X$ means p-incompressibility.

Let D be a central division F-algebra of degree a power p^n of a prime p. According to [9], for any $i = 0, \ldots, n$, the (generalized Severi-Brauer) variety $X(p^i; D)$ (of the right ideals in D of the reduced dimension p^i) is p-incompressible. Moreover, the p-motive of the variety $X(1; D)$ (this is the usual Severi-Brauer variety of D) is indecomposable.

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In the case of the prime $p = 2$, these results have important consequences for orthogonal involutions on central simple algebras, cf. [10]. For a similar study of unitary involutions, one would need similar results on Weil transfers (with respect to separable quadratic field extensions) of generalized Severi-Brauer varities. Since a central simple algebra A over a separable quadratic field extension L/F admits a unitary (F-linear) involution if and only if the norm algebra $N_{L/F}A$ is trivial, [14, Theorem 3.1(2)], only the case of a trivial norm algebra is of interest from this viewpoint.

The main result of this paper is the following theorem, where we write $\mathcal{R}_{L/F}X$ for the Weil transfer of an L-variety X.

Theorem 1.1. *Let F be a field, L/F a quadratic separable field extension, n a non-negative integer, and D a central division L-algebra of degree 2^n such that the norm algebra $N_{L/F}D$ is trivial. For any integer $i \in [0,\ n]$, the variety $\mathcal{R}_{L/F}X(2^i; D)$ is 2-incompressible.*

Moreover, in the case of a usual Severi-Brauer variety we also have the 2-motivic indecomposability:

Theorem 1.2. *In the settings of Theorem 1.1, the 2-motive of the variety $\mathcal{R}_{L/F}X(1; D)$ is indecomposable.*

As explained above, the choice of settings for Theorems 1.1 and 1.2 is mainly motivated by possible applications to unitary involutions. Here are some more arguments motivating this choice.

The assumption that the field extension L/F is separable (which, in particular, insures that the Weil transfer of a projective variety is again a projective variety) allows us to use the motivic Weil transfer functor constructed in [8].

We do not consider extensions L/F of degree > 2. The case of a quadratic extension seems to be the natural case to start with. Moreover, the answer in the case of $[L : F] > 2$ should depend on many initial parameters (like relations between the conjugate algebras of D). At the same time, a separable extension of degree > 2 can be non-galois, which makes the picture even more complicated. One needs a really strong motivation, which we do not have now, to attack such a situation.

Since we stay with quadratic extensions, we do not consider canonical p-dimension (and p-incompressibility) for odd primes p. Indeed, since cdim_p is not changed under finite p-coprime field extensions by [15, Proposition 1.5], the canonical p-dimension of the Weil transfer $\mathcal{R}(X)$ of any L-variety X coincides – if the prime p is odd – with the canonical p-dimension of the L-variety $\mathcal{R}(X)_L$ which is isomorphic to the product of X by its conjugate. Therefore, the problem of computing the canonical p-dimension (still interesting in some cases and trivial in some others, like in the trivial norm algebra case, our main case of consideration here) has not much to do with the Weil transfer anymore and is a problem concerning canonical p-dimension of products. We refer to [13] where such a problem is addressed, partially solved, and applied.

Now let p be any prime, L/F an arbitrary finite separable field extension, and A an arbitrary central simple L-algebra. For any generalized Severi-Brauer variety X of A, the canonical p-dimension of $\mathcal{R}_{L/F}X$ can be easily computed in terms of $\operatorname{cdim}_p \mathcal{R}_{L/F}X(p^i; D)$, where D is a central division L-algebra Brauer-equivalent to the p-primary part of A and where i runs over the non-negative integers satisfying $p^i < \deg D$, see Lemma 5.1.

Therefore, Theorem 1.1 determies $\operatorname{cdim}_p \mathcal{R}_{L/F}X$ in the case of $p = 2 = [L : F]$ and of trivial $N_{L/F}A$.

Finally, the assumption on the norm of D made in Theorems 1.1 and 1.2, can be avoided in the usual Severi-Brauer variety case. It turns out that a generic field extension killing the norm algebra $N_{L/F}D$ does not affect the canonical 2-dimension of the variety $\mathcal{R}_{L/F}X(1; D)$, see Proposition 5.2. As a consequence, the following three conditions are equivalent:

(1) the variety $\mathcal{R}_{L/F}X(1; D)$ is 2-incompressible;
(2) the 2-motive of the variety $\mathcal{R}_{L/F}X(1; D)$ is indecomposable;
(3) the division algebra D remains division over the function field of the variety $X(1; N_{L/F}D)$.

In Section 2 we discuss the motivic functor given by the Weil transfer. In Section 3 we write down some simple or known motivic decompositions used afterwards. In Section 4, both Theorems 1.1 and 1.2 are proved. Some easy generalizations are considered in Section 5.

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2. Motivic Weil transfer

Let F be a field. We fix a quadratic separable field extension L/F and we write $\mathcal{R}_{L/F}X$ (or simply $\mathcal{R}X$) for the Weil transfer of an L-variety X (see [8] for the definition and basic properties of $\mathcal{R}$ as well as for further references on it). We are working with the category $\mathrm{CM}(F, \Lambda)$ (constructed – in contrast to [5] – out of smooth *projective*, not just complete, F-varieties) of the Chow F-motives with coefficients in an associative unital commutative ring Λ, [5, §64] (we will set $\Lambda = \mathbb{F}_2$, the field of 2 elements, in the next section). We recall that the Weil transfer extends to motives giving a (non-additive and not commuting with the Tate shift) functor $\mathrm{CM}(L, \Lambda) \to \mathrm{CM}(F, \Lambda)$ of the category of L-motives into the category of F-motives, [8]. We write $\operatorname{cor}_{L/F}$ (or simply cor) for the (additive and commuting with the Tate shift) functor $\mathrm{CM}(L, \Lambda) \to \mathrm{CM}(F, \Lambda)$, studied in [11], associating to the motive of an L-variety X the motive of the F-variety $\operatorname{cor} X$, which is is the scheme X considered as an F-variety via the composition $X \to \operatorname{Spec} L \to \operatorname{Spec} F$.

Finally, σ is the non-trivial automorphism of L/F; σX, the *conjugate* of X, is the base change of X by $\sigma : \operatorname{Spec} L \to \operatorname{Spec} L$, and $\sigma : \mathrm{CM}(L, \Lambda) \to \mathrm{CM}(L, \Lambda)$ is the induced motivic (conjugation) functor.

Lemma 2.1. *For any two L-motives M and N one has*

$$\mathcal{R}(M \oplus N) \simeq \mathcal{R}(M) \oplus \operatorname{cor}(M \otimes \sigma N) \oplus \mathcal{R}(N).$$

Besides, $\mathcal{R}(M \otimes N) \simeq \mathcal{R}(M) \otimes \mathcal{R}(N)$ and $\mathcal{R}(\mathbb{F}_2(i)) \simeq \mathbb{F}_2(2i)$ for any integer i.

Proof. The formulas for a tensor product and for a Tate motive are in [8, Theorem 5.4]. We only have to prove the first formula.

Let us start by assuming that M and N are the motives of some L-varieties X and Y. We recall that the F-variety $\mathcal{R}X$ is determined (up to an isomorphism) by the fact that there exists a σ-isomorphism (that is, an isomorphism commuting with the action of σ) of the L-varieties $(\mathcal{R}X)_L$ and $X \times \sigma X$. At the same time, the F-variety X, which we denote as $\operatorname{cor} X$, is determined (up to an isomorphism) by existence of a σ-isomorphism of the L-varieties $(\operatorname{cor} X)_L$ and $X \coprod \sigma X$.

Since the L-varieties

$$(X \coprod Y) \times \sigma(X \coprod Y) \quad \text{and} \quad (X \times X) \coprod (X \times \sigma Y) \coprod \sigma(X \times \sigma Y) \coprod (Y \times \sigma Y)$$

are σ-isomorphic, it follows that

$$\mathcal{R}(X \coprod Y) \simeq (\mathcal{R}X) \coprod (\operatorname{cor} X \coprod \sigma Y) \coprod (\mathcal{R}Y),$$

whence the motivic formula.

In the general case, we have $M = (X, [\pi])$ and $(Y, [\tau])$ for some algebraic cycles π and τ ($[\pi]$ and $[\tau]$ are their classes modulo rational equivalence). Using the same letter $\mathcal{R}$ also for the Weil transfer of algebraic cycles, defined in [8, §3], as well as for the Weil transfer of their classes, defined in [8, §4], and processing similarly with the notation cor and σ, we get the following formula to check:

$$\mathcal{R}([\pi] + [\tau]) = \mathcal{R}[\pi] + \operatorname{cor}([\pi] + \sigma[\tau]) + \mathcal{R}[\tau].$$

This formula is easy to check because it holds already on the level of algebraic cycles, that is, with $[\pi]$ and $[\tau]$ replaced by π and τ. Since the group of algebraic cycles of any F-variety injects into the group of algebraic cycles of the same variety considered over L, it suffices to check the latter formula over L, where it becomes the trivial relation

$$(\pi + \tau) \cdot \sigma(\pi + \tau) = \pi \cdot \sigma\pi + \pi \cdot \sigma\tau + \tau \cdot \sigma\pi + \tau \cdot \sigma\tau. \qquad \square$$

3. Some motivic decompositions

Starting from this section, we are working with Chow motives with coefficients in the finite field $\mathbb{F}_2$, that is, we set $\Lambda = \mathbb{F}_2$ in the notation of the previous section. Therefore the Krull-Schmidt principle holds for the motives of the projective homogeneous varieties (see [3] or [9]). This means that any summand of the motive of a projective homogeneous variety possesses a finite direct sum decomposition with indecomposable summands and such a decomposition (called *complete* in this paper) is unique in the usual sense.

First we recall several known facts about the motives of Severi-Brauer varieties.

Let F be a field, D a central simple F-algebra, and let X be the Severi-Brauer variety $X(1; D)$ of D. If D is a division algebra, then the motive $M(X)$ of X is indecomposable. The original proof of this fact is in [6], a simpler recent proof can be found in [9].

Now, let us assume that D is the algebra of (2×2)-matrices over a central simple F-algebra C and set $Y = X(1; C)$. Then, according to [6], the motive of X decomposes in a sum of shifts of the motive of Y, namely,

$$M(X) \simeq M(Y) \oplus M(Y)(\deg C).$$

(This is, of course, a particular case of a general formula on the case where D is the algebra of $(r \times r)$-matrices over C for some $r \geq 2$.)

Finally, for arbitrary D, let D' be one more central simple F-algebra (we will have $\deg D' = \deg D$ in the application), $X' := X(1; D')$, and assume that the class of D' in the Brauer group $\operatorname{Br} F$ belongs to the subgroup generated by the class of D. Then the projection $X \times X' \to X$ is a projective bundle and therefore

$$M(X \times X') = M(X) \otimes M(X') \simeq \bigoplus_{i=0}^{\dim X'} M(X)(i)$$

by the motivic projective bundle theorem, [5].

A known consequence of this decomposition is as follows. Assume that D and D' are division algebras whose classes generate the same subgroup in $\operatorname{Br} F$ (for instance, D' can be the *opposite* algebra of D). Then $M(X \times X')$ is also expressed in terms of $M(X')$ and it follows by the Krull-Schmidt principle that $M(X) \simeq M(X')$. A different proof, working in a more general case of a generalized Severi-Brauer variety is given shortly below (in the last paragraph before Lemma 3.1).

Now let us describe the similar results concerning the *generalized* Severi-Brauer varieties. Since we are only interested in the 2-primary algebras in this paper and for the sake of simplicity, we assume that D is a central simple F-algebra of degree 2^n with some $n \geq 0$. Let $X = X(2^i; D)$ with some i satisfying $0 \leq i \leq n$. If D is division, then the variety X is 2-incompressible (though the motive of X is usually *decomposable* for $i \neq 0, 1, n$ by [17]). In motivic terms, the 2-incompressibility of X is expressed as follows: the indecomposable *upper* summand M_X of $M(X)$ is *lower*. The adjective *upper*, introduced in [9], simply means that the 0-codimensional Chow group of M_X is non-zero. By the Krull-Schmidt principle, the motive M_X is unique up to an isomorphism. The adjective *lower*, also introduced in [9], means that the d-dimensional Chow group of M_X is non-zero, where $d = \dim X$. This notion is dual to the notion of upper: the dual of an upper summand is lower and vice versa. Therefore, the 2-incompressibility of X also means that the summand M_X is self-dual.

Now let D and D' be central simple F-algebras whose classes in $\operatorname{Br} F$ generate the same subgroup. Let M and M' be the upper indecomposable motivic summands of the varieties $X := X(r; D)$ and $X' := X(r'; D')$, where r and r' are integers satisfying $0 \leq r \leq \deg D$, $0 \leq r' \leq \deg D'$, and $\gcd(r, \operatorname{ind} D) = \gcd(r', \operatorname{ind} D')$. Then $X(F(X')) \neq \emptyset \neq X'(F(X))$, and it follows by [9] that $M \simeq M'$.

Lemma 3.1. *Fix integers i and n satisfying $0 \leq i \leq n-1$. Let D be a central division F-algebra of degree 2^n and let U be the upper indecomposable motive of the variety $X(2^i; D)$. Let K/F be a field extension and C a central division K-algebra such that D_K is isomorphic to the algebra of (2×2)-matrices over C. For any integer j with $0 \leq j \leq n-1$, let V_j be the upper indecomposable motive of the K-variety $X(2^j; C)$ and $V := V_i$. Then the complete motivic decomposition of U_K contains V and $V(2^{i+n-1})$, while each of the remaining summands is a shifts of V_j with some $j < i$.*

Proof. Let $X = X(2^i; D)$ and $Y = X(2^i; C)$. Note that there exist rational maps in both directions between the K-varieties X_K and Y. Therefore the upper indecomposable motivic summands of the varieties Y and X_K are isomorphic. Since V is upper indecomposable and U_K is upper, V is a summand of U_K. Since U and V are self-dual (they

are self-dual because the varieties X and Y are 2-incompressible), we get by dualizing that $V(-\dim Y)$ is a summand of $U_K(-\dim X)$, that is, $V(\dim X - \dim Y)$ is a summand of U_K. Since $\dim X = 2^i(2^n - 2^i)$ and, similarly, $\dim Y = 2^i(2^{n-1} - 2^i)$, we have $\dim X - \dim Y = 2^{i+n-1}$.

According to [7, Corollary 10.19] (more general results of [4] or of [2] can be used here instead), the motive of X_K decomposes in a sum over the integers $r \in [0,\ 2^i]$ of the motives of the products $X(r;C) \times X(2^i - r;C)$ with some shifts. According to [9], the complete motivic decomposition of the product $X(r;C) \times X(2^i - r;C)$ consists of shifts of the motives V_j with $2^j|r$. A shift of V appears only two times: one time for $r = 0$ and another time for $r = 2^i$. Since the remaining indecomposable summands of $M(X_K)$ are shifts of $V(j)$ with $j < i$, the remaining indecomposable summands of U_K are also shifts of $V(j)$ with $j < i$. □

Lemma 3.2. *For i, n, D, and U as in Lemma 3.1, let U_j with $0 \le j \le n$ be the upper indecomposable motivic summand of $X(2^j;D)$. The complete decomposition of the tensor product $U \otimes U$ contains U and $U(d)$, where $d = 2^i(2^n - 2^i) = \dim X(2^i;D)$, while the remaining summands are $U(j)$ with some $j \in [1,\ d-1]$ and shifts of U_j with some $j < i$.*

Proof. Let $X = X(2^i;D)$. Since there exist rational maps in both directions between the varieties $X \times X$ and X, U is an upper indecomposable summand of $M(X \times X)$. Since $U \otimes U$ is an upper summand of $M(X \times X)$, it follows that U is a summand of $U \otimes U$. Dualizing, we get that $U(d)$ is a summand of $U \otimes U$.

According to [9], the complete motivic decomposition of $X \times X$ consists of shifts of U_j with $j \le i$. Since the summand U is upper while $U(d)$ is lower, a summand $U(j)$ with some j is present among the remaining summands of $M(X \times X)$ only if $j \in [1,\ d-1]$. Since $U \otimes U$ is a summand of $M(X \times X)$, the same statement holds for $U \otimes U$. □

4. Proofs of the main theorems

We recall that we are working with the Chow motives with coefficients in the finite field $\mathbb{F}_2$, that is, $\Lambda = \mathbb{F}_2$ in the notation of Section 2.

We start proving Theorem 1.2:

Proof of Theorem 1.2. We set $X := X(1;D)$.

We induct on n. For $n = 0$ we have $X = \operatorname{Spec} L$, $\mathcal{R}X = \operatorname{Spec} F$, and the statement is trivial. Below we are assuming that $n > 0$ and that the statement holds for all fields and all central division algebras of degree 2^{n-1}.

Let M be an upper motivic summand of $\mathcal{R}X$ (the adjective *upper* is defined in Section 3). It suffices to show that M is the whole motive of $\mathcal{R}X$.

Our proof is illustrated by Figure 1. An explanation of the illustration is given right after the end of the proof.

We recall that the L-variety $(\mathcal{R}X)_L$ is isomorphic to $X \times \sigma X$, where σX is the conjugate variety. Note that $\sigma X = X(1;\sigma D)$, where σD is the conjugate algebra. Since $(N_{L/F}D)_L \simeq D \otimes_L \sigma D$ and the F-algebra $N_{L/F}D$ is trivial, the L-algebra σD is opposite to D. Therefore, as explained in Section 3, the complete motivic decomposition of the

L-variety $(\mathcal{R}X)_L \simeq X \times \sigma X$ looks as follows (note that $\dim X = 2^n - 1$):

$$M(\mathcal{R}X)_L \simeq \bigoplus_{i=0}^{2^n-1} M(X)(i).$$

Since the summand M is upper, the complete decomposition of M_L contains a summand isomorphic to $M(X)$. (The Krull-Schmidt principle is used at this point and will be used several times later on in the proof.)

Let E be the function field of the variety $\mathcal{R}X(2^{n-1}; D)$. We write EL for the field $E \otimes_F L$. The EL-algebra $D \otimes_F E = D \otimes_L (EL) = D_{EL}$ is isomorphic to the algebra of (2×2)-matrices over a central division EL-algebra C of degree 2^{n-1}. By Section 3, the motive of the EL-variety X_{EL} decomposes as $N \oplus N(2^{n-1})$, where N is the motive of $X(1; C)$. Therefore the complete motivic decomposition of the EL-variety $(\mathcal{R}X)_{EL}$ consists of the summands $N(i)$, where the integer i runs over the interval $[0,\ 2^n + 2^{n-1} - 1]$. More precisely, there are two copies of $N(i)$ for $i \in [2^{n-1},\ 2^n - 1] \subset [0,\ 2^n + 2^{n-1} - 1]$ and one copy for $i \in [0,\ 2^n + 2^{n-1} - 1] \setminus [2^{n-1},\ 2^n - 1]$.

Since M_L contains $M(X)$, M_{EL} contains N.

We are working with the fields of the diagram

$$\begin{tikzcd} & EL \arrow[dl, dash] \arrow[dr, dash] & \\ L \arrow[dr, dash] & & E \arrow[dl, dash] \\ & F & \end{tikzcd}$$

Recall that the motive of the EL-variety X_{EL} decomposes as $N \oplus N(2^{n-1})$. Therefore, by Lemma 2.1, the motive of the E-variety $(\mathcal{R}X)_E = \mathcal{R}_{EL/E}(X_{EL})$ decomposes as

$$\mathcal{R}N \oplus \mathrm{cor}(N \otimes N')(2^{n-1}) \oplus \mathcal{R}N(2^n),$$

where $\mathcal{R} = \mathcal{R}_{EL/E}$, $\mathrm{cor} = \mathrm{cor}_{EL/E}$, and where N' is the conjugate of N.

The motive $\mathcal{R}N$ is indecomposable by the induction hypothesis. Since

$$N \otimes N' \simeq \bigoplus_{i=0}^{2^{n-1}-1} N(i)$$

(by Section 3) and the functor cor preserves indecomposability by [11], we get the complete motivic decomposition of $(\mathcal{R}X)_E$ replacing the summand $\mathrm{cor}(N \otimes N')(2^{n-1})$ by the sum $\bigoplus_{i=2^{n-1}}^{2^n-1} \mathrm{cor}\, N(i)$.

Since M_{EL} contains N, M_E contains $\mathcal{R}N$ and it follows that M_{EL} contains $N(i)$ for $i = 0, 1, \ldots, 2^{n-1} - 1$. Looking at the complete motivic decomposition of $(\mathcal{R}X)_L$, we see that M_L contains $M(X)(i)$ for such i and therefore M_{EL} also contains $N(2^{n-1} + i)$.

It follows that M_E contains $\mathrm{cor}\, N(i)$ for $i = 2^{n-1}, \ldots, 2^n - 1$. Since $(\mathrm{cor}\, N)_{EL} \simeq N \oplus N' \simeq N \oplus N$ (note that $N' \simeq N$ by Section 3), M_{EL} contains both copies of $N(i)$ for such i. We conclude that M_L contains $M(X)(i)$ also for $i = 2^{n-1}, \ldots, 2^n - 1$. Consequently, $M_L = M(\mathcal{R}X)_L$. It follows that $M = M(\mathcal{R}X)$ and Theorem 1.2 is proved. $\square$

Figure 1 illustrates the proof, just finished, in the case of $n = 3$. The ovals represent the summands of a complete motivic decomposition of the variety $(\mathcal{R}X)_{EL}$. Note that two

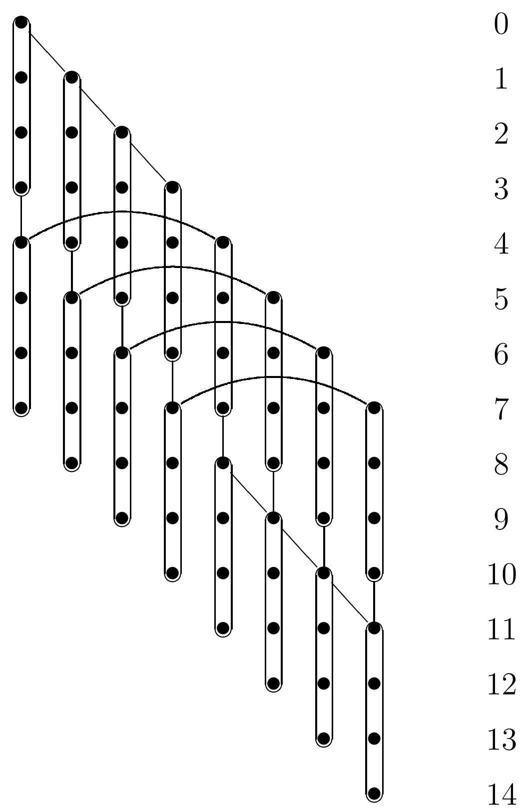

FIGURE 1. Proof of Theorem 1.2

different complete motivic decompositions of $(\mathcal{R}X)_{EL}$ have been used in the proof: one is a refinement of a complete motivic decomposition of $(\mathcal{R}X)_L$, the other of $(\mathcal{R}X)_E$. But because of the Krull-Schmidt principle, the sets of summands of these two decompositions can be identified in such a way that the identified summands are isomorphic. All the summands are shifts of N: there is one copy of $[N(i)]$ for $i = 0, 1, 2, 3, 8, 9, 10, 11$ and two copies of $[N(i)]$ for $i = 4, 5, 6, 7$. Over an algebraic closure of EL, each summand $N(i)$ becomes the sum of the Tate motives $\mathbb{F}_2(j)$ with $j = i, i+1, i+2, i+3$; these are indicated by the bullets inside of the ovals (the numbers on the right indicating the shifts of the corresponding Tate motives). The connection lines between the ovals are there to show that the connected ovals are inside of the same indecomposable summand over a smaller field: the vertical lines are the connections coming from the field L, while the others are from the field E with the straight lines coming from $\mathcal{R}N$ due to the induction hypothesis and the curved lines coming from $\operatorname{cor} N$. Note that the identification mentioned above (which arises from the Krull-Schmidt principle) does not preserves connections. So, it is a lucky coincidence that all pairs of isomorphic summands in each of two decompositions are E-connected (otherwise we would not be able to decide which of two isomorphic summands to use when drawing the L-connections; because of the coincidence the choice does not matter). Since all the ovals turn out to be connected by (a chain of) connection lines, and F is a subfield of both L and E, the motive of the F-variety $\mathcal{R}X$ is indecomposable.

We come to the proof of Theorem 1.1 now:

Proof of Theorem 1.1. We fix the integer $i \geq 0$ and we induct on $n \geq i$. For $n = i$ the statement is trivial. Below we are assuming that $n > i$ and that the statement (with the fixed i) holds for all fields and all central division algebras of degree 2^{n-1}.

Let U be the upper (see Section 3) indecomposable motivic summand of the variety $X(2^i; D)$. Since this variety is 2-incompressible, the summand U is lower (see Section 3). Note that by Lemma 2.1, $\mathcal{R}U$ is a motivic summand of the variety $\mathcal{R}X(2^i; D)$. Let M be a summand of $\mathcal{R}U$ which is upper as a motivic summand of $\mathcal{R}X(2^i; D)$. It suffices to show that M is lower.

Our proof is illustrated by Figure 2 right after the end of the proof. An explanation of the illustration is given in the end of the proof.

The L-variety $\big(\mathcal{R}X(2^i; D)\big)_L$ is isomorphic to $X(2^i; D) \times X(2^i; \sigma D)$, and the conjugate algebra σD is opposite to D. Therefore the upper indecomposable motivic summand of this variety is isomorphic to U and the lower indecomposable motivic summand of this variety is isomorphic to $U\big(\dim X(2^i; D)\big) = U\big(2^i(2^n - 2^i)\big)$. Since the summand M is upper, M_L contains U. Our aim is to show that M_L contains $U\big(2^i(2^n - 2^i)\big)$.

As in the proof of Theorem 1.2, we write E for the function field of the variety $\mathcal{R}X(2^{n-1}, D)$ and we are working with the diagram of fields

$$\begin{tikzcd}[column sep=small] & EL \arrow[dl, dash] \arrow[dr, dash] & \\ L \arrow[dr, dash] & & E \arrow[dl, dash] \\ & F & \end{tikzcd}$$

The EL-algebra D_{EL} is isomorphic to the algebra of (2×2)-matrices over a central division EL-algebra C of degree 2^{n-1}. By Lemma 3.1, the motive U_{EL} decomposes as $V \oplus V(2^{i+n-1}) \oplus \dots$, where V is the upper motive of the variety $X(2^i; C)$ and $\dots$ stands for a sum of the upper motives of varieties $X(2^j; C)$ with $j < i$. By Lemma 3.2, if $i < n-1$, the tensor product $V \otimes V$ decomposes as $V \oplus V(d) \oplus ? \oplus \dots$, where $d = \big(2^i(2^{n-1} - 2^i)\big)$, ? is a sum of $V(j)$ with $j \in [1,\ d-1]$, and where $\dots$ stands for a sum of the upper motives of varieties $X(2^j; C)$ with $j < i$. Therefore the complete motivic decomposition of $(\mathcal{R}U)_{EL}$ looks as

$$V \oplus V(d) \oplus V(2^{i+n-1}) \oplus V(2^{i+n-1} + d) \oplus V(2^{i+n-1}) \oplus V(2^{i+n-1} + d)$$
$$\oplus V(2^{i+n}) \oplus V(2^{i+n} + d) \oplus ? \oplus \dots.$$

Here ? is a sum of $V(j)$ with j in the (disjoint) union of the intervals

$$[1,\ d-1] \cup [2^{i+n-1} + 1,\ 2^{i+n-1} + d - 1] \cup [2^{i+n} + 1,\ 2^{i+n} + d - 1].$$

(The three intervals are pairwise disjoint because $d < 2^{i+n-1}$.) And, as before, $\dots$ stands for a sum of the upper motives of varieties $X(2^j; C)$ with $j < i$.

In the case of $i = n-1$, we have $d = 0$, V is the Tate motive $\mathbb{F}_2$, and $V \otimes V = V$. Therefore each of the pairs of summands $V(r)$, $V(r+d)$ for $r = 0, 2^{i+n-1}$ (two times), 2^{i+n} in the complete motivic decomposition of $(\mathcal{R}U)_{EL}$ indicated above is replaced by a single $V(r)$ (and the sum ? is empty).

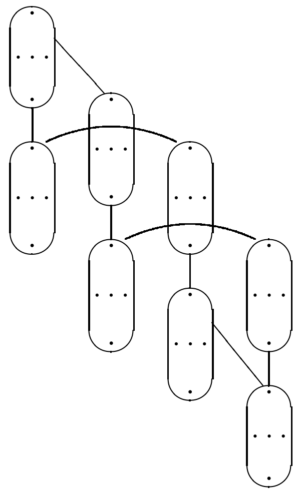

FIGURE 2. Proof of Theorem 1.1

By now we know that M_{EL} contains V and (at least one copy of) $V(2^{i+n-1})$.

Recall that U_{EL} decomposes as $V \oplus V(2^{i+n-1}) \oplus \dots$. Therefore, by Lemma 2.1, $(\mathcal{R}U)_E = \mathcal{R}_{EL/E} U_{EL}$ decomposes as

$$\mathcal{R}V \oplus \mathrm{cor}(V \otimes V)(2^{i+n-1}) \oplus \mathcal{R}V(2^{i+n}) \oplus \dots .$$

Any upper summand of $\mathcal{R}V$ is lower by the induction hypothesis. It follows that M_{EL} contains $V(d)$. Therefore M_{EL} contains (at least one copy of) $V(2^{i+n-1}+d)$. It follows that M_E contains $\mathrm{cor}\, V(2^{i+n-1}+d)$. Therefore M_{EL} contains both copies of $V(2^{i+n-1}+d)$ and it finally follows that M_{EL} contains $V(2^{i+n}+d)$ and is lower. $\square$

Figure 2 illustrates the proof, just finished (in the case of $d > 0$). The ovals represent some summands in a complete motivic decomposition of the variety $(\mathcal{R}X)_{EL}$: one copy of $[V]$, $[V(d)]$, $[V(2^{i+n})]$, $[V(2^{i+n}+d)]$ and two copies of $[V(2^{i+n-1})]$ and of $[V(2^{i+n-1}+d)]$. None of the remaining summand of the decomposition is isomorphic to a represented one. Note that two *different* complete motivic decompositions of the variety $\mathcal{R}X(2^i; D)$ over the field EL have been used in the proof: one is a refinement of a complete motivic decomposition over L, the other over E. But the sets of summands of these decompositions can be identified because of the Krull-Schmidt principle.

In contrast with Figure 1, where the summands are "thin", those of Figure 2 are, in general, "thick", that is, contain (over an algebraic closure of EL) several copies of the Tate motives with a same shift number. Since the picture illustrates the general case, we cannot (and do not) indicate the single Tate motives inside of the ovals. We only take care of representing a summand with a bigger shift number by a lower position oval. The connection lines between the ovals are there to show that the connected ovals are inside of the same indecomposable summand over a smaller field: the vertical lines are the connections coming from the field L, while the others are from the field E with the straight lines coming from $\mathcal{R}V$ due to the induction hypothesis and the curved lines coming from the $\mathrm{cor}\, V$. Note that the mentioned above identification (which arises from the Krull-Schmidt principle) does not preserves connections. So again, it is a lucky coincidence

that all pairs of isomorphic summands in each of two decompositions are E-connected (otherwise we would not be able to decide which of two isomorphic summands to use when drawing the L-connections).

Since the upper oval turns out to be connected by (a chain of) connection lines with the lower one, the variety $\mathcal{R}X$ is 2-incompressible.

5. Generalizations

Let p be any prime, L/F an arbitrary finite separable field extension, and A an arbitrary central simple L-algebra. For any integer r with $0 \leq r \leq \deg A$, the canonical p-dimension of $\mathcal{R}_{L/F}X(r; A)$ can be easily computed in terms of $\operatorname{cdim}_p \mathcal{R}_{L/F}X(p^i; D)$, where D the central division L-algebra Brauer-equivalent to the p-primary part of A and where i runs over the non-negative integers satisfying $p^i < \deg D$:

Lemma 5.1. *In the above settings, we have*

$$\operatorname{cdim}_p \mathcal{R}_{L/F}X(r; A) = \operatorname{cdim}_p \mathcal{R}_{L/F}X(p^i; D),$$

where $i = \min\{v_p(r),\ v_p(\deg D)\}$ *with* $v_p(\cdot)$ *standing for the* p*-adic order.*

Proof. The variety $\mathcal{R}_{L/F}X(p^i; D)$ has a point over the function field of $\mathcal{R}_{L/F}X(r; A)$:

$$\Big(\mathcal{R}_{L/F}X(p^i; D)\Big)\Big(F\big(\mathcal{R}_{L/F}X(r; A)\big)\Big) = \\ X(p^i; D)\Big(L\big(X(r; A)\big) \otimes_L L\big(X(r; \sigma A)\big)\Big) \supset X(p^i; D)\Big(L\big(X(r; A)\big)\Big) \neq \emptyset$$

because $\operatorname{ind} D_{L(X(r;A))}$ divides p^i. Similarly, the variety $\mathcal{R}_{L/F}X(r; A)$ has a point over a finite p-coprime extension of the function field of $\mathcal{R}_{L/F}X(p^i; D)$. □

We turn back to the case of $p = 2 = [L : F]$ in order to remove the norm condition of Theorems 1.1 and 1.2 in the case of a usual Severi-Brauer variety. Note that the function field $F(Y)$ in the following statement is a (generic) splitting field of the norm algebra of D:

Proposition 5.2. *Let L/F be a quadratic separable field extension, D a 2-primary central division L-algebra, $X = \mathcal{R}_{L/F}X(1; D)$, and $Y = X(1; N_{L/F}D)$. Then* $\operatorname{cdim}_2 X = \operatorname{cdim}_2 X_{F(Y)}$.

Proof. Note that $X_{F(Y)} \simeq \mathcal{R}_{L(Y)/F(Y)}X(1; D_{L(Y)})$ and $Y_L \simeq X(1; D \otimes_L D')$, where D' is the conjugate algebra σD. According to the index reduction formula of [16], $\operatorname{ind} D_{L(Y)}$ is equal to the minimum of $\operatorname{ind}\big(D^{\otimes(i+1)} \otimes (D')^{\otimes i}\big)$ where i runs over the integers. Let i be an integer which gives the minimum and let $\tilde{D}$ be a central division L-algebra Brauer-equivalent to the product $D^{\otimes(i+1)} \otimes (D')^{\otimes i}$. We set $\tilde{X} = \mathcal{R}X(1; \tilde{D})$. Since the algebra $\tilde{D}' := \sigma\tilde{D}$ is Brauer-equivalent to the product $(D')^{\otimes(i+1)} \otimes D^{\otimes i}$ and the exponent of D (coinciding with the exponent of D') is a power of 2, the classes of $\tilde{D}$ and $\tilde{D}'$ in $\operatorname{Br}(L)$ generate the same subgroup as the classes of D and D'. It follows that

$$\tilde{X}\big(F(X)\big) \neq \emptyset \neq X\big(F(\tilde{X})\big).$$

Therefore, for any field extension E/F, we have $\operatorname{cdim}_2 \tilde{X}_E = \operatorname{cdim}_2 X_E$. On the other hand, for $\tilde{Y} = X(1; N\tilde{D})$, the algebra $\tilde{D}_{L(\tilde{Y})}$ is division. Consequently, by Theorem 1.1,

the variety $\tilde{X}_{F(\tilde{Y})}$ is 2-incompressible. Therefore the variety $\tilde{X}$ is also 2-incompressible, $\operatorname{cdim}_2 \tilde{X} = \dim \tilde{X} = \operatorname{cdim}_2 \tilde{X}_{F(\tilde{Y})}$, and we obtain that $\operatorname{cdim}_2 X = \operatorname{cdim}_2 X_{F(\tilde{Y})}$. Since the norm algebra ND becomes trivial over $F(\tilde{Y})$, the field extension $F(\tilde{Y})(Y)/F(\tilde{Y})$ is purely transcendental, and the required statement follows. □

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UPMC Univ Paris 06, Institut de Mathématiques de Jussieu, F-75252 Paris, FRANCE

Web page: www.math.jussieu.fr/~karpenko
E-mail address: karpenko *at* math.jussieu.fr